

Duality symmetry for star products

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Abstract

A duality property for star products is exhibited. In view of it, known star-product schemes, like the Weyl-Wigner-Moyal formalism, the Husimi and the Glauber-Sudarshan maps are revisited and their dual partners elucidated. The tomographic map, which has been recently described as yet another star product scheme, is considered. It yields a noncommutative algebra of operator symbols which are positive definite probability distributions. Through the duality symmetry a new noncommutative algebra of operator symbols is found, equipped with a new star product. The kernel of the new star product is established in explicit form and examples are considered.

Introduction

Star-products, originally introduced in the context of the quantization problem, have recently attracted new interest partly because of their appearance in Noncommutative Geometry. Among the different star-product schemes known in the literature since many years, the Weyl-Wigner-Moyal formalism [1, 2, 3], which consists of an invertible map from (Hilbert-Schmidt) operators onto (Schwartzian) functions on \mathbb{R}^{2n} with the Moyal product, plays a distinguished role. Various modifications of such type of maps exist yielding new operator symbols, analogous to Wigner functions, known in the literature as the Husimi Q-function [4], the P-function introduced in [5, 6] and s-ordered quasi-probability distributions [7]. In a modern language we recognize these sets of symbols as noncommutative associative algebras equipped with a noncommutative (star) product.

As anticipated above, the interest in star products, which have been intensively studied for long time, has recently received new impulse (a very partial list of references is [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]). One of the reasons is certainly the emergence of Noncommutative Geometry in relevant sectors of physics like field theory, where it came out that the dynamics of strings in the presence of a magnetic field is described by a noncommutative geometry associated to the Moyal product [18]. Since then, field theories on Moyal-type spaces have been scrutinized extensively; good reviews are [19, 20]. The new products are most often either products of finite matrices (the fuzzy algebras [21]) or can be reduced to the Moyal product. Recently, [22] a general method is proposed, which produces various new non-formal star products on \mathbb{R}^3 using a variation of the Jordan Schwinger map [23]. The method includes previous results of [24] and it is easily extendable to higher dimensions [25].

In a different setting it was recently established [26, 27] that the symplectic [28, 29] and spin [30, 31] tomography, which furnish alternative formulations of quantum mechanics and quantum field theory [33], can be described as well within a star-product scheme. Moreover, in [27] different known star-product schemes were presented in a unified form. There, the symbols of the operators are defined in terms of a special family of operators using the trace formula (what we sometimes call the ‘dequantization’ map because of its original meaning in the Wigner-Weyl formalism), while the reconstruction of operators in terms of their symbols (the ‘quantization’ map) is determined using another family of operators. These two families determine completely the star-product scheme, including the kernel of the star-product.

The two families of operators allow to express any of the operators belonging to the space of linear operators as a combination of them. Moreover, they are dual to each other. Indeed, the key observation of the present article is that their role can be exchanged, without violating the consistency of the scheme. This implies that the family of operators originally used in the reconstruction map (from functions to operators) can be used instead in the ‘dequantization’ map to determine a different set of symbols which is a different noncommutative associative algebra, dual to the original one, with a new star product.

This duality symmetry is not specific to the tomographic map. The latter is just an example where it produces a new noncommutative algebra, whose nature is still to be fully understood. It is instead a property of the star-product scheme as it appears in the unifying form described in [27]. The symmetry could yield to interesting results in other relevant cases as those considered in [22, 25], which we plan to investigate later.

The paper is organized as follows. In sec. 1 we review the general building scheme for the star-product of operator symbols using two families of basic operators. In sec. 2 we introduce the duality symmetry and the dual star product. In section 3 we investigate the duality property of known star-product schemes. The Weyl-Wigner-Moyal formalism will result to be self-dual, as already known in various forms, while the so called s-ordered symbols will show a duality relation between normal-ordered (s=1) and antinormal-ordered (s=-1) symbols. In section 4 we review the tomographic map (symplectic tomography) and the star-product construction of the operator-tomograms. In Sec. 5 we introduce the dual tomographic map and study its properties. Moreover, we derive the kernel of the star-product for the dual tomographic symbols. Although the noncommutative algebra of symbols and the related star-product were already established, the dual algebra and the new star-product, discovered thanks to the duality symmetry, were not known to our knowledge. Conclusions and perspectives are given in Sec. 6.

1 The star product

In this section we review a general construction for the symbols of operators acting in a Hilbert space \mathcal{H} . We follow the presentation and the notation of [27].

Given an operator \hat{A} acting on the Hilbert space \mathcal{H} (which can be finite or infinite-dimensional), let us have two basic families $\hat{U}(\bar{x})$ and $\hat{D}(\bar{x})$ acting in \mathcal{H} . The families are labelled by a vector $\bar{x} = (x_1, \dots, x_N)$. The parameters $x_k, (k = 1, \dots, N)$ can be continuous (real or complex) or discrete. The symbol $f_A(\bar{x})$ of the operator \hat{A} is the function depending on the vector \bar{x} defined by the formula

$$f_A(\bar{x}) = \text{Tr}(\hat{A}\hat{U}(\bar{x})). \quad (1.1)$$

We assume that the trace exists for all the parameters \bar{x} . The introduced function is understood as a generalized function. The second family of operators, $\hat{D}(\bar{x})$, serves to reconstruct the operator if one knows its symbol, that is

$$\hat{A} = \int f_A(\bar{x}) \hat{D}(\bar{x}) d\bar{x}. \quad (1.2)$$

We assume that an appropriate measure $d\bar{x}$ exists to make sense of the reconstruction formula. If there are discrete components x_k the integral (1.2) splits into a sum over the discrete components and an integration over the remaining continuous ones.

The symbols form an associative algebra endowed with a non-commutative (star) product inherited by the operator product. The associativity follows from the associativity of the operator product $\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$. This defines for the symbols the star product

$$f_A * f_B(\bar{x}) = f_{AB}(\bar{x}) = \int f_A(\bar{x}_1) f_B(\bar{x}_2) K(\bar{x}_1, \bar{x}_2, \bar{x}) d\bar{x}_1 d\bar{x}_2, \quad (1.3)$$

where the kernel K must satisfy the nonlinear equation

$$\int K(\bar{x}_1, \bar{y}, \bar{x}) K(\bar{x}_2, \bar{x}_3, \bar{y}) d\bar{y} = \int K(\bar{x}_1, \bar{x}_2, \bar{y}) K(\bar{y}, \bar{x}_3, \bar{x}) d\bar{y} \quad (1.4)$$

obtained by imposing the associativity condition. Let us get the expression of the kernel of the star product in terms of the operators $\hat{U}(\bar{x})$ and $\hat{D}(\bar{x})$. To obtain it we take (1.2) for an operator \hat{A} and multiply it on the right by an analogous expression for an operator \hat{B} . Then we calculate the symbol of this product by means of (1.1) and we obtain

$$f_{AB}(\bar{y}) = \int \text{Tr} \left(f_A(\bar{x}) \hat{D}(\bar{x}) f_B(\bar{x}') \hat{D}(\bar{x}') \hat{U}(\bar{y}) \right) d\bar{x} d\bar{x}'. \quad (1.5)$$

Comparing this expression with (1.3) we have

$$K(\bar{x}_1, \bar{x}_2, \bar{y}) = \text{Tr}(\hat{D}(\bar{x}_1) \hat{D}(\bar{x}_2) \hat{U}(\bar{y})). \quad (1.6)$$

Because of the construction of the kernel and the associativity of the operator product we are guaranteed that (1.6) satisfies the nonlinear equation (1.4).

Writing the formula (1.1) in view of (1.2) one has

$$f_A(\bar{x}) = \text{Tr} \left\{ \left[\int f_A(\bar{x}') \hat{D}(\bar{x}') d\bar{x}' \right] \hat{U}(\bar{x}) \right\} \quad (1.7)$$

and, assuming we can exchange the trace with the integral, we arrive at the equality

$$f_A(\bar{x}) = \int d\bar{x}' \left[\text{Tr} \hat{D}(\bar{x}') \hat{U}(\bar{x}) \right] f_A(\bar{x}'). \quad (1.8)$$

The compatibility condition for Eqs. (1.1)-(1.8) requires then

$$\text{Tr} \hat{D}(\bar{x}') \hat{U}(\bar{x}) = \delta(\bar{x}' - \bar{x}), \quad (1.9)$$

where δ is to be replaced with the Kronecker delta for discrete parameters. Going back to (1.6), it is to be stressed that the kernel of the star product has been obtained solely in terms of the operators $D(\bar{x})$ and $U(\bar{x})$, which in turn are only constrained by (1.9). Thus, to each pair of operators satisfying (1.9) it is associated a associative algebra with a star product. The role of $D(\bar{x})$ and $U(\bar{x})$ can be exchanged, what gives rise to a duality symmetry.

2 The dual map and the dual star-product

The two families of operators $\hat{U}(\bar{x})$ and $\hat{D}(\bar{x})$ are connected by the equation (1.9). This implies that they can be interpreted as orthogonal families in the space of operators. In fact, it is well known that the linear structure in the space of operators (matrices) allows to consider them as vectors (see for example [32]). The trace of two operators is equivalent then to the standard scalar product in the linear vector space. In this context the symbol of an operator defined by means of the family \hat{U} can be treated as a projection onto the ‘vector’ $\hat{U}_{\bar{x}} \equiv \hat{U}(\bar{x})$. But the ‘vectors’ $\hat{D}_{\bar{x}} \equiv \hat{D}(\bar{x})$ provide a dual family since their scalar product with the vectors $\hat{U}_{\bar{x}}$ is equal to a delta function. In view of this geometrical picture it is obvious that one can define a dual symbol $f_A^{(d)}(\bar{x})$ of an operator \hat{A} using the projection on the dual family $\hat{D}_{\bar{x}}$, i.e.

$$f_A^{(d)}(\bar{x}) = \text{Tr}(\hat{A}\hat{D}(\bar{x})). \quad (2.1)$$

The reconstruction formula for the operator \hat{A} reads then

$$\hat{A} = \int f_A^{(d)}(\bar{x}) \hat{U}(\bar{x}) d\bar{x} \quad (2.2)$$

which means that we have exchanged the role of \hat{U} and \hat{D} in our previous formulae (1.1) and (1.2). This duality property being simple, provides new nontrivial symbols and a new associated star product with a new kernel in all cases where the operators \hat{U} and \hat{D} are essentially different. From Eq. (1.6) the expression for the kernel of the dual star product is easily derived to be

$$K^{(d)}(\bar{x}_1, \bar{x}_2, \bar{y}) \equiv \text{Tr} \left(\hat{U}(\bar{x}_1) \hat{U}(\bar{x}_2) \hat{D}(\bar{y}) \right). \quad (2.3)$$

3 The Weyl-Wigner and the s-ordered maps

In this section we review known examples of quantization-dequantization maps from the viewpoint of the duality property illustrated above.

Given an operator \hat{A} acting in the Hilbert space of square integrable functions $\psi(x)$ on \mathbb{R} , the Wigner-Weyl symbol of \hat{A} and the reconstruction map are defined by means of the families of operators

$$\hat{U}(\bar{x}) = 2\hat{\mathcal{D}}(\alpha)(-\hat{1})^{a^\dagger a} \mathcal{D}(-\alpha) \quad (3.1)$$

$$\hat{D}(\bar{x}) = \frac{1}{2\pi} \hat{U}(\bar{x}) \quad (3.2)$$

where $(-\hat{1})^{a^\dagger a}$ is the parity operator. We may use a symplectic notation for the real vector and write $\bar{x} = (q, p)$. The annihilation and creation operators a and a^\dagger read

$$a = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}), \quad a^\dagger = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}). \quad (3.3)$$

The unitary displacement operator (complex Weyl system) realizing the ray representation of the group of translations of the plane is of the form

$$\hat{\mathcal{D}}(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \quad (3.4)$$

with $\alpha = (q + ip)/\sqrt{2}$. Using known properties of this operator

$$(-\hat{1})^{a^\dagger a} \hat{\mathcal{D}}(\alpha) (-\hat{1})^{a^\dagger a} = \hat{\mathcal{D}}(-\alpha) \quad (3.5)$$

$$\text{Tr } \hat{\mathcal{D}}(\alpha) = \pi \delta(\text{Re } \alpha) \delta(\text{Im } \alpha) \quad (3.6)$$

one can check that

$$\text{Tr } \hat{U}_{\bar{x}} \hat{D}_{\bar{x}'} = \delta(q - q') \delta(p - p'). \quad (3.7)$$

The Weyl symbol of the operator \hat{A} reads

$$f_A(q, p) = \text{Tr}(\hat{U}(\bar{x}) \hat{A}) \quad (3.8)$$

which in coordinates representation becomes

$$f_A(q, p) = \int A(q + \frac{u}{2}, p - \frac{u}{2}) e^{-ipu} du \quad (3.9)$$

with $A(x, x') = \langle x | \hat{A} | x' \rangle$. For the density operator $\hat{\rho}$ of a normalized state the Weyl symbol is exactly what is known as the Wigner function of the quantum state

$$W(q, p) = \int \rho(q + \frac{u}{2}, p - \frac{u}{2}) e^{-ipu} du. \quad (3.10)$$

Moreover the Weyl symbol of the identity operator is $f_1(q, p) = 1$ where 1 is the identity in the algebra of functions on \mathbb{R}^2 , whereas the Weyl symbols of the position and momentum operators are $f_q(q, p) = q$, $f_p(q, p) = p$. The operator \hat{A} can be expressed in terms of its Weyl symbol by means of the reconstruction formula (1.2)

$$\hat{A} = \frac{1}{\pi} \int f_A(q, p) \hat{\mathcal{D}}(\alpha) (-\hat{1})^{a^\dagger a} \hat{\mathcal{D}}(-\alpha) dq dp \quad (3.11)$$

which, specialized to the density operator, yields the well known relation between the density operator and the Wigner function. In coordinates representation it reads

$$\rho(x, x') = \frac{1}{2\pi} \int W(\frac{x+x'}{2}, p) e^{ip(x-x')} dp. \quad (3.12)$$

The algebra of symbols is what is known as the Moyal plane, that is a noncommutative algebra of functions on \mathbb{R}^2 with the Moyal product. The kernel is easily obtained specializing (1.6) to this case and using Eq. (3.5) together with the product rule for the displacement operators

$$\hat{\mathcal{D}}(\alpha) \hat{\mathcal{D}}(\beta) = \mathcal{D}(\alpha + \beta) e^{i \text{Im}(\alpha \beta^*)} \quad (3.13)$$

Thus we reduce the calculation of the kernel to evaluate a trace of the form (3.6). The final result may be written in the form

$$K(q_1, p_1, q_2, p_2, q_3, p_3) = \frac{1}{\pi^2} \exp \{2i [q_1(p_2 - p_3) + q_2(p_3 - p_1) + q_3(p_1 - p_2)]\} \quad (3.14)$$

or equivalently

$$K(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{\pi^2} \exp \{4i \text{Im} [(\alpha_1^* \alpha_2 + \alpha_3 \alpha_2^* + \alpha_1 \alpha_3^*)]\}. \quad (3.15)$$

As from (3.1), (3.2), in the Wigner-Weyl-Moyal formalism the operators $\hat{U}(\bar{x})$ and $\hat{D}(\bar{x})$ coincide up to a numerical factor, which implies that this scheme is selfdual. That is, the duality operation doesn't produce a new algebra of symbols nor a new star-product.

3.1 s-ordered symbols

In [7] s-ordered quasidistribution functions which are s-ordered symbols of density operators were introduced as a generalization of the Wigner, Husimi and Glauber-Sudarshan distribution functions. Later on it has been realized that these symbols, together with their reconstruction formulas, may be obtained through the two families of operators $\hat{U}_s(x)$ and $\hat{D}_s(x)$, $\bar{x} = (q, p)$ of the form

$$\hat{U}_s(x) = \frac{2}{(1-s)} \hat{D}(\alpha) \left(\frac{s+1}{s-1} \right)^{a^\dagger a} \hat{D}(-\alpha) \quad (3.16)$$

$$\hat{D}_s(x) = \frac{1}{2\pi} \hat{U}_{-s}(x) \quad (3.17)$$

where s is a real parameter. Moreover s-ordered symbols may be considered not only for the density operator but for a generic operator \hat{A} according to the general scheme presented in the previous sections. The case $s = 0$ corresponds to the standard *Wigner – Weyl* situation described above. In the limit $s = \pm 1$ we have respectively the diagonal representation of the density matrix (P-function) by Sudarshan [5] and Glauber [6] and the Q-quasidistribution [4]. The star-product kernel of s-ordered symbols is calculated along the same lines than the Moyal kernel (3.15), thus yielding [27]

$$K(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{(1-s^2)\pi^2} \exp \left\{ \frac{2}{1+s} (\alpha_1^* \alpha_2 - \alpha_2 \alpha_3^* - \alpha_3 \alpha_1^*) + \frac{2}{1-s} (\alpha_3 \alpha_2^* + \alpha_1 \alpha_3^* - \alpha_2^* \alpha_1) + \frac{4s}{s^2-1} |\alpha^3|^2 \right\} \quad (3.18)$$

to be compared with (3.15). It is important to stress that the two limiting cases corresponding to $s = \pm 1$ are in the duality relation discussed previously. In particular, the Husimi Q-function and the P-function are dual symbols of the density operator in such limit cases. They belong to two different associative noncommutative algebras, with different star-products. It is already known that the two quantum schemes correspond to

normal and antinormal ordering of the creation and annihilation operators. The duality symmetry then connects the two orderings. The Wigner-Weyl Moyal quantization scheme selects instead the symmetric ordering, consistently with its being selfdual.

Another simple example of a consistent pair of operators $\hat{U}(\bar{x}), \hat{D}(\bar{x})$ producing a noncommutative algebra and a star product in the sense of (1.1), (1.3), is provided by the pair

$$\hat{U}(\bar{x}) = \frac{1}{\sqrt{2\pi}} e^{i\mu\hat{q} + i\nu\hat{p}}, \quad \hat{D}(\bar{x}) = \hat{U}^\dagger(\bar{x}) \quad (3.19)$$

with $\bar{x} = (\mu, \nu)$, μ, ν , real. This construction amounts to consider the Fourier transform of the Weyl symbol of the operator \hat{A} . That is, the symbol $f_A(\mu, \nu) = \text{Tr}(\hat{U}(\bar{x})\hat{A})$ is related to the Weyl symbol as

$$f_A(\mu, \nu) = \frac{1}{\sqrt{2\pi}} \int f_A(q, p) e^{i\mu q + i\nu p} dq dp. \quad (3.20)$$

The reconstruction formula reads instead

$$\hat{A} = \frac{1}{\sqrt{2\pi}} \int e^{-i\mu\hat{q} - i\nu\hat{p}} f_A(\mu, \nu) d\mu d\nu. \quad (3.21)$$

It is evident from Eq. (3.19) that the dual star-product construction corresponds to the replacement $\mu, \nu \rightarrow -\mu, -\nu$ or, from Eq. (3.20), to the replacement $f_A(q, p) \rightarrow f_A(-q, -p)$. The kernel of the star-product for the introduced symbols can be calculated as the trace of the product of three exponents

$$K(\mu_1, \nu_1, \mu_2, \nu_2, \mu_3, \nu_3) = \frac{1}{(\sqrt{2\pi})^3} \text{Tr}(e^{i\mu_3\hat{q} + i\nu_3\hat{p}} e^{-i\mu_1\hat{q} - i\nu_1\hat{p}} e^{-i\mu_2\hat{q} - i\nu_2\hat{p}}) \quad (3.22)$$

which yields

$$K(\mu_1, \nu_1, \mu_2, \nu_2, \mu_3, \nu_3) = (2\pi)^{3/2} \delta(\mu_1 + \mu_2 - \mu_3) \delta(\nu_1 + \nu_2 - \nu_3) \exp \left[\frac{i}{2} (\mu_3\nu_1 - \mu_1\nu_3 + \mu_3\nu_2 - \mu_2\nu_3 + \mu_2\nu_1 - \mu_1\nu_2) \right]. \quad (3.23)$$

4 The tomographic map

The tomographic map was originally introduced in [28, 29] to solve an old issue in quantum mechanics, namely to provide a description of quantum mechanics which were directly in terms of probability distributions (see also [33, 34]). Because of its original scope the map with its inverse was in the beginning proposed only for the density operators, as it was the case for the distribution functions of Wigner, Glauber-Sudarshan and Husimi.¹ It was immediately clear that the space of distributional functions associated to density operators

¹In such context it is also referred as the *probability representation* of quantum mechanics because the quantum states are described by families of probabilities.

(the density matrix tomograms) is not a commutative one but it took some time to realize that it could be described within a star-product scheme [26, 27]. In [26, 27] it was also understood that the scheme is actually more general, that is, not only density operators but a generic operator can be mapped into tomograms with a well defined associative star-product, thus exhibiting a new example of noncommutative algebra which might be interesting *per se*, independently from its original connection to quantum mechanics.

For the tomographic map the two families of operators $\hat{U}(x)$ and $\hat{D}(x)$ have been shown to be

$$\hat{U}(\bar{x}) = \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}) \quad (4.1)$$

$$\hat{D}(\bar{x}) = \frac{1}{2\pi} \exp [i(X\hat{1} - \mu\hat{q} - \nu\hat{p})] \quad (4.2)$$

where $\bar{x} = (X, \mu, \nu) \in \mathbb{R}^3$. Thus one can prove [35] that the tomographic symbol of the density operator, originally introduced in [28, 29] in the form

$$\mathcal{W}_\rho(X, \mu, \nu, t) = \frac{1}{2\pi} \int dk e^{-ikX} \text{Tr}(\hat{\rho} e^{ik\hat{X}}), \quad (4.3)$$

with $X = (\mu q + \nu p)$, $\hat{X} = (\mu\hat{q} + \nu\hat{p})$, may be reexpressed by means of Eqs. (1.1), (4.1) as

$$\mathcal{W}_\rho(X, \mu, \nu) = \text{Tr}(\hat{\rho} \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})) \quad (4.4)$$

where we are adopting the notation \mathcal{W}_A for the tomographic symbols. From this formula it may be directly checked that \mathcal{W}_ρ is a probability density, that is

$$\mathcal{W}_\rho \geq 0, \quad \text{and} \quad \int \mathcal{W}_\rho(X, \mu, \nu) dx = 1.$$

According to Eq. (1.3) the star product of two tomographic symbols $\mathcal{W}_A, \mathcal{W}_B$ is

$$\begin{aligned} \mathcal{W}_A * \mathcal{W}_B(X, \mu, \nu) &= \int dX_1 d\mu_1 d\nu_1 dX_2 d\mu_2 d\nu_2 \\ &\mathcal{W}_A(X_1, \mu_1, \nu_1) \mathcal{W}_B(X_2, \mu_2, \nu_2) K(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2, X, \mu, \nu) \end{aligned} \quad (4.5)$$

where the kernel K , defined in Eq. (1.6) is given by

$$\begin{aligned} K(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2, X, \mu, \nu) &= \text{Tr}(\hat{D}(X_1, \mu_1, \nu_1) \hat{D}(X_2, \mu_2, \nu_2) \hat{U}(X, \mu, \nu)) \\ &= \frac{1}{4\pi^2} \exp \left\{ \frac{i}{2} \left[(\nu_1 \mu_2 - \mu_1 \nu_2 + 2X_1 + 2X_2) - \left(\frac{\nu_1 + \nu_2}{\nu} + \frac{\mu_1 + \mu_2}{\mu} \right) X \right] \right\} \\ &\delta(\mu(\nu_1 + \nu_2) - \nu(\mu_1 + \mu_2)). \end{aligned} \quad (4.6)$$

In order to verify the compatibility condition (1.9) we can calculate the trace of the product of the operators $\hat{U}(\bar{x})$ and $\hat{D}(\bar{x})$. It reads

$$\begin{aligned} \text{Tr}[\hat{U}(\bar{x}) \hat{D}(\bar{x}')] &= \frac{1}{2\pi} e^{iX'} \text{Tr} \left[e^{-i\mu' \hat{q} - i\nu' \hat{p}} \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}) \right] \\ &= \frac{1}{2\pi} \delta(\nu\mu' - \mu\nu') \exp \left\{ \frac{i}{2} \left[2X' - \left(\frac{\mu'}{\mu} + \frac{\nu'}{\nu} \right) X \right] \right\} \end{aligned} \quad (4.7)$$

Although expressed in unusual form, this is a three dimensional delta function in the space of functions of (X, μ, ν) . It can be easily verified observing that

$$\int f(X', \mu', \nu') \frac{1}{2\pi} \delta(\nu\mu' - \mu\nu') \exp \left\{ \frac{i}{2} \left[2X' - \left(\frac{\mu'}{\mu} + \frac{\nu'}{\nu} \right) X \right] \right\} dX' d\mu' d\nu' = f(X, \mu, \nu). \quad (4.8)$$

5 Dual tomographic map

In this section we address the problem of constructing the dual tomographic map by means of the duality symmetry previously exploited. Following the prescription, the new families of basic operators \hat{U} and \hat{D} are obtained exchanging the role of (4.1) and (4.2), that is, we have now

$$\hat{U}_d(\bar{x}) = \frac{1}{2\pi} \exp [i (X\hat{1} - \mu\hat{q} - \nu\hat{p})] \quad (5.1)$$

$$\hat{D}_d(\bar{x}) = \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}). \quad (5.2)$$

This means that we associate with the operator \hat{A} the symbol

$$\mathcal{W}_A^{(d)}(X, \mu, \nu) = \frac{1}{2\pi} e^{iX} \text{Tr}(\hat{A} e^{-i\mu\hat{q} - i\nu\hat{p}}). \quad (5.3)$$

One can see that this function differs from the symbol determined by Eq. (3.20) just by the factor $[\exp(iX)]/\sqrt{2\pi}$. The reconstruction formula for the dual tomographic map reads

$$\begin{aligned} \hat{A} &= \frac{1}{2\pi} \int e^{iX} \left(\text{Tr} \hat{A} e^{-i\mu\hat{q} - i\nu\hat{p}} \right) \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}) dX d\mu d\nu \\ &= \int \mathcal{W}_A^{(d)}(X, \mu, \nu) \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}) dX d\mu d\nu. \end{aligned} \quad (5.4)$$

Integrating over dX we obtain

$$\hat{A} = \frac{1}{2\pi} \int \left[\text{Tr} \hat{A} e^{-i\mu\hat{q} - i\nu\hat{p}} \right] e^{i\mu\hat{q} + i\nu\hat{p}} d\mu d\nu \quad (5.5)$$

which is exactly Eq. (3.21).

The dual tomographic symbols satisfy the constraint

$$\frac{\partial}{\partial X} \left(e^{-iX} \mathcal{W}_A^{(d)}(X, \mu, \nu) \right) = 0 \quad (5.6)$$

namely the dependence on X is factorized and it is just a phase.

Now we can calculate the kernel of the star product as

$$K^{(d)}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{1}{2\pi} e^{iX_3} \text{Tr} \left[\delta(X_1\hat{1} - \mu_1\hat{q} - \nu_1\hat{p}) \delta(X_2\hat{1} - \mu_2\hat{q} - \nu_2\hat{p}) \delta(X_3\hat{1} - \mu_3\hat{q} - \nu_3\hat{p}) \right] \quad (5.7)$$

Using the Fourier decomposition for the delta function we arrive at

$$K^{(d)}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{1}{4\pi^2|S_{12}|} \exp\left(\frac{i}{S_{12}}V - \frac{i}{2}\frac{S_{23}S_{31}}{S_{12}}\right) \quad (5.8)$$

where the volume V and the symplectic areas S_{ik} read respectively

$$V = \bar{X}[\bar{\mu} \times \bar{\nu}] \quad (5.9)$$

$$S_{ij} = \mu_i \nu_j - \mu_j \nu_i \quad (5.10)$$

and $\bar{X} = (X_1, X_2, X_3)$, $\mu = (\mu_1, \mu_2, \mu_3)$, $\nu = (\nu_1, \nu_2, \nu_3)$. The kernel for the $*$ commutator of dual tomographic symbols is then

$$\begin{aligned} \mathcal{K}^{(d)}(\bar{x}_1, \bar{x}_2, \bar{x}_3) &= K^{(d)}(\bar{x}_1, \bar{x}_2, \bar{x}_3) - K^{(d)}(\bar{x}_2, \bar{x}_1, \bar{x}_3) \\ &= \frac{i}{2\pi^2|S_{12}|} \exp\left(\frac{iV}{S_{12}}\right) \sin\left(\frac{S_{23}S_{31}}{2S_{12}}\right) \end{aligned} \quad (5.11)$$

which in turn defines a Poisson bracket on the algebra of dual tomograms, if the commutative limit is performed. We will deepen this analysis elsewhere.

5.1 Examples

In order to better understand what is the dual tomographic map it is worth establishing the form of dual tomographic symbols for some relevant operators. From Eq. (5.3) we have for the identity operator

$$\mathcal{W}_1^{(d)}(X, \mu, \nu) = e^{iX} \delta(\mu) \delta(\nu). \quad (5.12)$$

This is the identity in the space of dual tomograms. Analogously we find for powers of momentum and position operators

$$\mathcal{W}_{q^n}^{(d)}(X, \mu, \nu) = (i)^n e^{iX} \delta^{(n)}(\mu) \delta(\nu) \quad (5.13)$$

$$\mathcal{W}_{p^n}^{(d)}(X, \mu, \nu) = (i)^n e^{iX} \delta(\mu) \delta^{(n)}(\nu) \quad (5.14)$$

where $\delta^{(n)}$ is the n -th derivative of the Dirac delta function.

The density operator $|\alpha\rangle\langle\alpha|$ of the coherent state $|\alpha\rangle$, is mapped to

$$\mathcal{W}_\alpha^{(d)}(X, \mu, \nu) = \frac{e^{iX}}{2\pi} e^{-\frac{(\mu^2 + \nu^2)}{4}} \exp\left[i\text{Im}\left(\alpha^* \frac{(\nu - i\mu)}{\sqrt{2}}\right)\right] \quad (5.15)$$

while the density operator $|n\rangle\langle n|$ of the Fock state $|n\rangle$ corresponds to

$$\mathcal{W}_n^{(d)}(X, \mu, \nu) = \frac{e^{iX}}{2\pi} e^{-\frac{(\mu^2 + \nu^2)}{4}} L_n\left(\frac{\mu^2 + \nu^2}{2}\right) \quad (5.16)$$

where L_n are Laguerre polynomials; also, the dual symbol of the transition operator $|n\rangle\langle m|$ has the form

$$\mathcal{W}_{m,n}^{(d)}(X, \mu, \nu) = \frac{e^{iX}}{2\pi} e^{-\frac{(\mu^2 + \nu^2)}{4}} \left(\frac{\nu - i\mu}{\sqrt{2}} \right)^{m-n} L_n^{m-n} \left(\frac{\mu^2 + \nu^2}{2} \right). \quad (5.17)$$

For comparison, we report the symbols related to coherent and Fock states in the standard tomographic map. They are respectively

$$\mathcal{W}_\alpha(X, \mu, \nu) = \frac{1}{\sqrt{\pi(\mu^2 + \nu^2)}} \exp \left[-\frac{(X - \sqrt{2} \mu \operatorname{Re} \alpha - \nu \operatorname{Im} \alpha)^2}{\mu^2 + \nu^2} \right] \quad (5.18)$$

and

$$\mathcal{W}_n(X, \mu, \nu) = \frac{1}{\sqrt{\pi(\mu^2 + \nu^2)}} \exp \left[-\frac{X^2}{\mu^2 + \nu^2} \right] \frac{1}{2^n n!} H_n^2 \left(\frac{X}{\sqrt{\mu^2 + \nu^2}} \right) \quad (5.19)$$

where H_n are Hermite polynomials.

6 Conclusions

To conclude we point out the main results of the work. The duality symmetry of the star product scheme described by Eqs. (1.1), (1.2), which provides dual algebras of operator symbols was elucidated. Some known results which give raise to special families of symbols relevant for quantum mechanics, like the quasi-distribution functions of Husimi (Q-functions) and Glauber-Sudarshan (P-quasi-distributions) are shown to be connected via the duality symmetry. The main result of the paper is the construction of the dual tomographic map and the explicit calculation of the kernel for the dual star-product. This yields a new noncommutative algebra, the algebra of dual tomographic symbols.

References

- [1] H. Weyl *The theory of groups and Quantum Mechanics*, Dover (1931).
- [2] E. Wigner, Phys. Rev. **40**, 749 (1932).
- [3] H. Grönwold, Physica **12**, 405 (1946).
J. E. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949).
- [4] K. Husimi, Proc. Phys. Math. Soc. Jpn. **23**, 264 (1940).
- [5] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
- [6] R. J. Glauber, Phys. Rev. Lett. **10**, 84 (1963).
- [7] K. Cahill and R. Glauber, Phys. Rev. **177**, 1882 (1969).

- [8] F. A. Berezin Math. USSR-Izv. **8**, 1109 (1974).
- [9] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Ann. Phys. (NY) **111**, 61 (1978); Ann. Phys. (NY) **111**, 111 (1978).
- [10] J.C. Varilly and J. M. Gracia-Bondia Ann. Phys. (N. Y.), **190**, 107 (1989).
- [11] J. M. Gracia-Bondia Cont. Math. **134**,93 (1992).
- [12] V.V. Dodonov, V. I. Man'ko and D. L. Ossipov, in *Theory of Nonstationary quantum oscillators*, M.A. Markov ed. Proc. Lebedev Phys. Inst. **198**, 1-58 (1992)
- [13] B. Fedosov *Deformation quantization and index theory*, Mathematical topics **9** Akademie Verlag, Berlin (1996).
- [14] M. Kontsevich, [arXiv:math.QA/9709040].
- [15] T. Curtright, T. Uematsu and C. K. Zachos, J. Math. Phys. **42** (2001) 2396 [arXiv:hep-th/0011137].
- [16] M. V. Karasev and T. A. Osborn J. Math. Phys. **43**, 756 (2002).
- [17] M. Daoud Phys. Lett. **A309**, 167 (2003)
- [18] J. Frohlich and K. Gawedzki, CRM Proc. Lecture Notes **7**, 57 (1994) [arXiv:hep-th/9310187].
G. Landi, F. Lizzi and R. J. Szabo, Commun. Math. Phys. **206**, 603 (1999) [arXiv:hep-th/9806099].
N. Seiberg, E. Witten, JHEP **9909**, 032 (1999).
- [19] M.R. Douglas and N. A. Nekrasov, Rev.Mod.Phys. **73**, 977 (2001) [arXiv:hep-th/0106048].
- [20] R.J. Szabo, Phys.Rept. **378** 207, (2003) [arXiv:hep-th/0109162].
- [21] J. Madore, Class. Quant. Grav. **9**, 69 (1992).
H. Grosse, C. Klimcik and P. Presnajder, Commun. Math. Phys. **180** (1996) 429 [arXiv:hep-th/9602115].
A. P. Balachandran, B. P. Dolan, J. H. Lee, X. Martin and D. O'Connor, J. Geom. Phys. **43**, 184 (2002) [arXiv:hep-th/0107099].
G. Alexanian, A. Pinzul and A. Stern, Nucl. Phys. B **600**, 531 (2001) [arXiv:hep-th/0010187].
F. Lizzi, P. Vitale and A. Zampini, JHEP **0308**, 057 (2003) [arXiv:hep-th/0306247];
Mod. Phys. Lett. A **18**, 2381 (2003) [arXiv:hep-th/0309128].
- [22] J. M. Gracia-Bondia, F. Lizzi, G. Marmo, P. Vitale; JHEP **0204**, 026 (2002) [arXiv:hep-th/0112092].
- [23] V.I. Man'ko, G. Marmo, P. Vitale and F. Zaccaria, Int. J. Mod. Phys. **A9**, 5541 (1994) , [arXiv:hep-th/9310053].
- [24] A.B. Hammou, M. Lagraa and M.M. Sheikh-Jabbari, Phys.Rev. **D66**, 025025 (2002) [arXiv:hep-th/0110291].
- [25] A. Agostini, F. Lizzi and A. Zampini, Mod. Phys. Lett. A **17**, 2105 (2002) [arXiv:hep-th/0209174].
- [26] O.V. Man'ko, V. I. Man'ko, G. Marmo, Physica Scripta **62**, 446 (2000)

- [27] O.V. Man'ko, V. I. Man'ko, G. Marmo, Journal of Physics A **35**, 699 (2002)
- [28] S. Mancini, V. I. Man'ko, and P. Tombesi, Quantum Semiclass. Opt. **7**, 615 (1995).
- [29] S. Mancini, V. I. Man'ko, and P. Tombesi, Phys. Lett. **A213**, 1 (1996); Found. Phys. **27**, 801 (1997).
- [30] V.V. Dodonov and V. I. Man'ko Phys. Lett. **A 229**, 335 (1997).
- [31] V.I. Man'ko and O. V. Man'ko, JETP **85**, 430 (1997).
- [32] V. I. Man'ko, G. Marmo, E. C. G. Sudarshan, and F. Zaccaria, J. Russ. Las. Res. (Kluwer Academic/Plenum Publishers)**24**, 507 (2003).
- [33] V. I. Man'ko, L. Rosa, and P. Vitale, Phys. Lett. **B 439**, 328 (1998), [arXiv:hep-th/9806164].
- [34] V. I. Man'ko, L. Rosa, and P. Vitale, Phys. Rev. **A 57**, 3291(1998), [arXiv:quant-ph/9802030].
- [35] M. A. Man'ko, V. I. Man'ko, and R. Vilela-Mendez J. Phys. A **34**, 8321 (2001).
- [36] V. I. Man'ko, G. Marmo, E. C. G. Sudarshan, and F. Zaccaria, Phys. Lett. **A 273**, 31 (2000); J. Russ. Las. Res. (Kluwer Academic/Plenum Publishers) **20**, 421 (1999).